Math 126 Final Exam

UCB Summer 2025

August 14th, 2025

Remember that the exam is closed-book. You will have 1 hour to complete the exam. There are 4 questions. You may use the front and back of each page, and there will be plenty of space (If you need extra paper for any reason, please attach it via paperclips or staples). Please Write your full name and SID below. Good luck!

Name: Solutions

SID: ____________

Formulas:

$$\int_U \nabla \cdot V dx = \int_{\partial U} \eta \cdot V dS$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + w = 0$$

$$\dot{x}(t) = v$$

$$\frac{Du}{Dt} + w = 0$$

Homogeneous Solution: $u(t,x) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)d\tau$

$$\mathcal{D}_{t,x} = \{(s,y) \in \mathbb{R}^2 \mid x - c(t-s) \le y \le x + c(t-s)\}$$

Inhomogeneous Wave equation, no initial data solution: $u(t,x) = \frac{1}{2c}\int_{\mathcal{D}_{t,x}} f(s,y)dsdy$

Poisson's Formula: $u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{\mathbb{D}} \frac{g(x-ty)}{\sqrt{1-|y|^2}}dy\right) + \frac{t}{2\pi}\int_{\mathbb{D}} \frac{h(x-ty)}{\sqrt{1-|y|^2}}dy$

Heat Kernel: $H_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}}$

Bessel's Inequality: $\sum_{k\in\mathbb{Z}} |\langle f, \phi_k\rangle|^2 \leq \|f\|^2$

Series solution to the Heat Equation on $[0,\infty)\times\mathbb{T}$: $u(t,x) = \sum_{k\in\mathbb{Z}} c_k[h] e^{-kt^2} e^{ikx}$

Parseval's Identity : $\sum_{k\in\mathbb{Z}} |c_k[f]|^2 = \frac{1}{2\pi}\|f\|_{L^2}^2$

$$c_k[f] = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

Mean Value Formula:$u(x_0) = \frac{1}{A_n R^{n-1}}\int_{\partial B(x_0,R)} u(x)dS + \int_{B(x_0,R)} G_R(x-x_0)\Delta u(x)dx$

$$G_R(x) = \begin{cases} \frac{1}{2\pi}\ln(\frac{r}{R}) & n=2 \\ \frac{1}{(n-2)A_n}\left[\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right] & n\geq 3 \end{cases}$$

1 Problem 1

The following questions may be answered with true or false. You do not need to explain your answer, but please make your answer clear and box it for each part.

Part A) Let the Fourier coefficients of $f(x)$ be $c_k[f] = \frac{1}{k^3}$. Then, $f(x) \in C^1(\mathbb{T})$.

True: $\sum_{k \in \mathbb{Z}} k \cdot c_k[f] = \sum 1/k^2 < \infty.$

Part B) Let $H_0^1(\mathbb{R}) = \{f \in H^1(\mathbb{R}) \mid f(x) = \lim_{n\to\infty} p_n(x),\ p_n(x) \in C_c^\infty(\mathbb{R})\}$, so H_0^1 is the closure of the smooth functions in the H^1 norm. Then, H_0^1 is a Hilbert space.

True: A closed subspace of a complete space is complete

Part C) If we consider the 1-D heat equation on a metal rod of length l, and insulate the ends from heat, this corresponds to Dirichlet boundary conditions on the heat equation.

False: Neumann

Version B Note: Part A) here was i/k, So if $f'(x) \in C^0(\mathbb{T})$, $c_k[f] = 1/k^2$. Then, $c_k[f'(x)] = i/k$, $\left[\sum c_k[f'] e^{ikx}\right]_{x=0}$ should converge (Dirichlet-Jordan test). Since $\sum_{k \in \mathbb{Z}} i/k = \infty$, this fails and $f' \notin C^0$, so $f \notin C^1$.

This was harder than intended, so either True or false is acceptable.

2 Problem 2

Suppose that $U \subseteq \mathbb{R}^n$ is a bounded domain with $U \subseteq B(0, R)$ for some $R > 0$. Assume $u \in C^2(U:\mathbb{R}) \cap C^0(\overline{U}:\mathbb{R})$ satisfies

$$-\Delta u = f$$
$$u|_{\partial U} = 0$$

for some $f \in C^0(\overline{U})$ (so that f is bounded on U).

Part A) Find a constant $c > 0$ depending perhaps on f and R such that $u + c|x|^2$ is subharmonic on U.

For $c = \|f\|_\infty / 2n$, $-\Delta(u + c|x|^2) = f - \|f\|_\infty \leq 0$

So $u + c|x|^2$ is subharmonic.

Part B) Use this to show that

$$\max_{\overline{U}} |u| \leq C \max_{\overline{U}} |f|$$

(pay attention to the absolute value bars!)

$\max_{\bar{u}} u \leq \max_{\bar{u}} u + c|x|^2 \leq \max_{\partial u} (u + c|R|^2) = c \cdot R^2 = \|f\|_\infty \cdot \frac{R^2}{2n}$.

Similarly, $\min_{\bar{u}} u \geq \min_{\bar{u}} u - c|x|^2 \geq -\|f\|_\infty \cdot R^2/2n$.

For $d = R^2/2n$,

$\max_{\bar{u}} |u| \leq d \|f\|_\infty$.

3 Problem 3

Let $u \in C^2([0,\infty) \times U)$ for some bounded domain $U \subseteq \mathbb{R}^n$ satisfy

$$\int_0^\infty \int_U \left[u \frac{\partial^2 \psi}{\partial t^2} + \nabla u \cdot \nabla \psi \right] dx dt = - \int_U g \frac{\partial \psi}{\partial t}|_{t=0} dx + \int_U h\psi|_{t=0} dx \qquad (A).$$

for all $\psi \in C_c^\infty([0,\infty) \times U)$. Show that u solves the wave equation

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta) u = 0 \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

Assuming $u(0,x) = g(x)$, $\partial_t u(0,x) = h(x)$, and U is C^1,

$$\int_0^\infty \int_U u \frac{\partial^2 \psi}{\partial t^2} + \nabla u \cdot \nabla \psi \, dx dt$$

$$= \int_0^\infty \int_U (\partial_t^2 u - \Delta u) \psi \, dx dt + \int_U u \psi|_{t=0} dx - \int_U u \partial_t \psi|_{t=0} dx$$

Therefore, (A) holds iff. $\int_0^\infty \int_U [(\partial_t^2 - \Delta) u] \psi \, dx dt = 0$

for all $\psi \in C_c^\infty([0,\infty) \times U)$, so $(\partial_t^2 - \Delta) u = 0$.

• Without the initial condition assumptions, we have

$$\int_0^\infty \int_U [(\partial_t^2 - \Delta) u] \psi \, dx dt = - \int_U (g - u) \partial_t \psi|_{t=0} dx + \int_U (h - \partial_t u) \psi|_{t=0} dx$$

By considering $\psi \in C_c^\infty((0,\infty) \times U)$, $(\partial_t^2 - \Delta) u = 0$ in $(0,\infty) \times U$ and, by continuity, in $[0,\infty) \times U$.

Then, $- \int_U (g - u) \partial_t \psi|_{t=0} dx + \int_U (h - \partial_t u) \psi|_{t=0} dx = 0$.

Since $g, h \in L^1_{loc}(U)$, we may construct smooth bumps to show $h = \partial_t u|_{t=0}$, $g = u|_{t=0}$.

4 Problem 4

Recall that $\{\frac{1}{\sqrt{2\pi}}e^{ikx}\}$ is a basis for $L^2((-\pi,\pi))$.

Consider the domain $(0,\pi)$ and the orthonormal sequence $\psi_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$. Show that $\{\psi_k\}$ is an orthonormal basis for $L^2((0,\pi))$ (you only need to show that it is a basis, not that it is orthonormal). *Hint:* Given $f \in L^2((0,\pi))$, we may extend f to an odd function on $(-\pi,\pi)$. If $\int_{-\pi}^{\pi} f(x)\sin(kx)dx = 0$ for all k, what can you say about $\int_{-\pi}^{\pi} f(x)e^{ikx}dx$?

Let us assume $f(x)$ is such that $\int_0^{\pi} f(x)\overline{\psi_k(x)}dx = 0$ for all k. Since $\sin(kx)$ is odd, an odd extension of f to $(-\pi,\pi)$ has $\int_{-\pi}^{\pi} f(x)\sin(kx)dx = 2\int_0^{\pi} f(x)\sin(kx)dx = 0$

Notice also that $\int_{-\pi}^{\pi} f(x)e^{ikx}dx = \int_{-\pi}^{\pi} f(x)\cos(kx)dx + i\int_{-\pi}^{\pi} f(x)\sin(kx)dx$
$= 0$

Since $\{1/\sqrt{2\pi}\, e^{ikx}\}$ is a basis, this implies $f(x)=0$.

Therefore, if $\langle f(x), \psi_k(x)\rangle = 0$ for all k, $f(x)=0$.

Finally, $\{\psi_k(x)\}$ is an orthonormal basis.